Linear Algebra I

18/12/2015, Friday, 9:00 – 11:00

You are **NOT** allowed to use any type of calculators.

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1 $(2+6+4+4+4 = 20 \text{ pts})$ **Linear equations**

Consider the following linear system of equations

$$\begin{aligned} d + e - f &= 2 \\ a + 2b + e - f &= 0 \\ a + 2b + 2c - e + f &= 2. \end{aligned}$$

(a) Write down the augmented matrix.

(b) By performing row operations, put the augmented matrix into row echelon form.

(c) Determine the *lead* and *free* variables.

(d) By performing row operations, put the augmented matrix into row *reduced* echelon form.

(e) Find the solution set of the equation.

Required Knowledge: **Gauss-elimination, row operations, notions of lead/free variables.**

Solution:

1a: Augmented matrix is given by:

$$\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & \vdots & 2 \end{array}\right].$$

1b:

$$\left[\begin{array}{cccccccc} 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \\ 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 1 & 2 & 2 & 0 & -1 & 1 & \vdots & 2 \end{array}\right] \xrightarrow[\mathbf{2nd} = \mathbf{1st}]{\mathbf{1st} = \mathbf{2nd}} \left[\begin{array}{cccccccc} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \\ 1 & 2 & 2 & 0 & -1 & 1 & \vdots & 2 \end{array}\right]$$

$$\left[\begin{array}{cccccccc} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \\ 1 & 2 & 2 & 0 & -1 & 1 & \vdots & 2 \end{array}\right] \xrightarrow{\mathbf{3rd} = \mathbf{3rd} - \mathbf{1st}} \left[\begin{array}{cccccccc} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 2 & 0 & -2 & 2 & \vdots & 2 \end{array}\right]$$

$$\left[\begin{array}{cccccccc} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \\ 0 & 0 & 2 & 0 & -2 & 2 & \vdots & 2 \end{array}\right] \xrightarrow[\mathbf{3rd} = \mathbf{2nd}]{\mathbf{2nd} = \mathbf{3rd}} \left[\begin{array}{cccccccc} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 2 & 0 & -2 & 2 & \vdots & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \end{array}\right]$$

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 2 & 0 & -2 & 2 & \vdots & 2 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \end{bmatrix} \xrightarrow{\mathbf{2nd} = \frac{1}{2} \times \mathbf{2nd}} \begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \end{bmatrix}$$

1c: Lead variables are a, c, and d whereas b, e, and f are free variables.

1d: The matrix

$$\begin{bmatrix} 1 & 2 & 0 & 0 & 1 & -1 & \vdots & 0 \\ 0 & 0 & 1 & 0 & -1 & 1 & \vdots & 1 \\ 0 & 0 & 0 & 1 & 1 & -1 & \vdots & 2 \end{bmatrix}$$

is already in row reduced echelon form.

1e: The general solution is given by

$$\begin{aligned} a &= -2b - e + f \\ c &= e - f + 1 \\ d &= -e + f + 2 \end{aligned}$$

where b, e, and f are free variables.

Consider the matrix

$$\begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix}.$$

(a) Find the determinant.

(b) Determine all values of x for which this matrix is nonsingular.

REQUIRED KNOWLEDGE: **Determinants, nonsingular matrices.**

SOLUTION:

2a: By applying row operation type III, we can get

$$\det \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix} = \det \begin{pmatrix} 0 & 1-x & 1-x & 1-x^2 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 1 & 1 & x \end{pmatrix}.$$

Cofactor expansion along the last row results in

$$\det \begin{pmatrix} 0 & 1-x & 1-x & 1-x^2 \\ 0 & x-1 & 0 & 1-x \\ 0 & 0 & x-1 & 1-x \\ 1 & 1 & 1 & x \end{pmatrix} = -\det \begin{pmatrix} 1-x & 1-x & 1-x^2 \\ x-1 & 0 & 1-x \\ 0 & x-1 & 1-x \end{pmatrix}.$$

By applying row operation type II, we can get

$$\det \begin{pmatrix} 1-x & 1-x & 1-x^2 \\ x-1 & 0 & 1-x \\ 0 & x-1 & 1-x \end{pmatrix} = (1-x)^3 \det \begin{pmatrix} 1 & 1 & 1+x \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix}.$$

Note that

$$\det \begin{pmatrix} 1 & 1 & 1+x \\ -1 & 0 & 1 \\ 0 & -1 & 1 \end{pmatrix} = \det \begin{pmatrix} 0 & 1 \\ -1 & 1 \end{pmatrix} + \det \begin{pmatrix} 1 & 1+x \\ -1 & 1 \end{pmatrix} = 1 + 2 + x = 3 + x.$$

Therefore,

$$\det \begin{pmatrix} x & 1 & 1 & 1 \\ 1 & x & 1 & 1 \\ 1 & 1 & x & 1 \\ 1 & 1 & 1 & x \end{pmatrix} = -(1-x)^3(3+x) = (x-1)^3(3+x) = x^4 - 6x^2 + 8x - 3.$$

2b: A square matrix is nonsingular if and only if its determinant is nonzero. Therefore, the matrix we look at is nonsingular if and only if $(x-1)^3(3+x) \neq 0$, in other words, if and only if $x \neq 1$ and $x \neq -3$.

Let A, B, and C be $n \times n$ matrices and

$$M = \begin{bmatrix} A & B \\ C & 0_{n\times n} \end{bmatrix}.$$

(a) Show that M is nonsingular if and only if both B and C are nonsingular.

(b) Suppose that B and C are nonsingular. Find the inverse of M.

REQUIRED KNOWLEDGE: **Partitioned matrices and nonsingular matrices.**

SOLUTION:

3a: 'if': Suppose that both B and C are nonsingular. Let $z \in \mathbb{R}^{2n}$ be such that

$$Mz = 0.$$

Partition z as

$$z = \begin{bmatrix} x \\ y \end{bmatrix}$$

where $x, y \in \mathbb{R}^n$. Then, we have

$$0_{2n} = Mz = \begin{bmatrix} A & B \\ C & 0_{n\times n} \end{bmatrix} \begin{bmatrix} x \\ y \end{bmatrix} = \begin{bmatrix} Ax + By \\ Cx \end{bmatrix}.$$

This leads to

$$Ax + By = 0_n \quad \text{and} \quad Cx = 0_n.$$

Since C is nonsingular, we get $x = 0_n$ and hence $By = 0_n$. Now, it follows from non singularity of B that $y = 0_n$. Thus, we obtain that $z = 0_{2n}$. Consequently, M is nonsingular.

'only if': Suppose that M is nonsingular. Let $y \in \mathbb{R}^n$ be such that

$$By = 0_n.$$

Note that

$$M\begin{bmatrix} 0_n \\ y \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0_{n\times n} \end{bmatrix} \begin{bmatrix} 0_n \\ y \end{bmatrix} = \begin{bmatrix} By \\ 0_n \end{bmatrix} = 0_{2n}.$$

Since M is nonsingular, this means that $y = 0_n$. Therefore, the matrix B must be nonsingular. Similarly, let $x \in \mathbb{R}^n$ be such that

$$Cx = 0.$$

Note that

$$M\begin{bmatrix} x \\ -B^{-1}Ax \end{bmatrix} = \begin{bmatrix} A & B \\ C & 0_{n\times n} \end{bmatrix} \begin{bmatrix} x \\ -B^{-1}Ax \end{bmatrix} = \begin{bmatrix} Ax - Ax \\ Cx \end{bmatrix} = 0_{2n}.$$

Since M is nonsingular, this means that $x = 0$. Therefore, the matrix C must be nonsingular.

3b: Let the matrix

$$N = \begin{bmatrix} W & X \\ Y & Z \end{bmatrix}$$

be a candidate for the inverse. Note that

$$MN = \begin{bmatrix} A & B \\ C & 0_{n\times n} \end{bmatrix} \begin{bmatrix} W & X \\ Y & Z \end{bmatrix} = \begin{bmatrix} AW + BY & AX + BZ \\ CW & CX \end{bmatrix}.$$

In case N is the inverse of M, one has

$$\begin{aligned} AW + BY &= I_n \\ AX + BZ &= 0_{n\times n} \\ CW &= 0_{n\times n} \\ CX &= I_n. \end{aligned}$$

From the third equation, we obtain that $W = 0_{n\times n}$ as C is nonsingular. By substituting this into the first, we get

$$I_n = BY$$

and hence $Y = B^{-1}$. From the last and non singularity of C, we get $X = C^{-1}$. Finally, it follows from the second that $Z = -B^{-1}AC^{-1}$. Therefore, we obtain

$$M^{-1} = N = \begin{bmatrix} 0_{n\times n} & C^{-1} \\ B^{-1} & -B^{-1}AC^{-1} \end{bmatrix}.$$

Consider the vector space P_4. Let $S = \{p(x) \in P_4 \mid p(x) + p(-x) = 0\}$ and $L : P_4 \to P_4$ be given by $L(p(x)) = \frac{1}{2}(p(x) + p(-x))$.

(a) Are the vectors $1+x$, $x+x^2$, x^2+x^3, x^3+1 linearly independent?

(b) Are the vectors $1+x$, $x+x^2$, x^2+x^3, x^3 for a basis for P_4?

(c) Show that the set S is a subpace of P_4. Find a basis for S and determine its dimension.

(d) Show that L is a linear transformation.

(e) Find $\ker L$.

(f) Find the matrix representation of L with respect to the ordered basis $\{1+x, x+x^2, x^2+x^3, x^3\}$

Required Knowledge: **Subspaces, basis, dimension, linear transformations and their matrix representations.**

Solution:

4a: Let a, b, c, and d be scalars such that

$$a(1+x) + b(x+x^2) + c(x^2+x^3) + d(x^3+1) = 0.$$

This results in

$$(d+a) + (a+b)x + (b+c)x^2 + (c+d)x^3 = 0$$

and hence

$$\begin{aligned} d+a &= 0 \\ a+b &= 0 \\ b+c &= 0 \\ c+d &= 0. \end{aligned}$$

Since $(a,b,c,d) = (1,-1,1,-1)$ is a nontrivial solution for these equations, the vectors $1+x$, $x+x^2$, x^2+x^3, x^3+1 are linearly dependent.

4b: To form a basis, they need to be linearly independent and to span the vector space P_4. To check the former, let a, b, c, and d be scalars such that

$$a(1+x) + b(x+x^2) + c(x^2+x^3) + dx^3 = 0.$$

This leads to

$$a + (a+b)x + (b+c)x^2 + (c+d)x^3 = 0$$

and hence

$$\begin{aligned} a &= 0 \\ a+b &= 0 \\ b+c &= 0 \\ c+d &= 0. \end{aligned}$$

Clearly, the only solution for these equations is $a = b = c = d = 0$. As such, the vectors $1+x$, $x+x^2$, x^2+x^3, x^3 are linearly independent.

To check whether they span P_4, let $p(x)$ be an arbitrary polynomial belonging to P_4 given by

$$p(x) = \alpha + \beta x + \gamma x^2 + \delta x^3.$$

The question is if the polynomial p can be written as a linear combination of the vectors $1+x$, $x+x^2$, x^2+x^3, x^3; in other words, if we can find scalars a, b, c, and d such that

$$\alpha+\beta x+\gamma x^2+\delta x^3 = a(1+x)+b(x+x^2)+c(x^2+x^3)+dx^3.$$

This would result in

$$\begin{aligned} a &= \alpha \\ a+b &= \beta \\ b+c &= \gamma \\ c+d &= \delta. \end{aligned}$$

By solving these equations, we obtain $a=\alpha$, $b=\beta-\alpha$, $c=\gamma-\beta+\alpha$, and $d=\delta-\gamma+\beta-\alpha$. Therefore, the vectors $1+x$, $x+x^2$, x^2+x^3, x^3 span P_4. As they are already shown to be linearly independent, they form a basis for P_4.

4c: The set S is a subspace if it is nonempty and closed under vector addition and scalar multiplication:

- Clearly, $0 \in S$. So, the set S is nonempty.
- Let $p(x) \in S$ and a be a scalar. Note that $ap(x)+ap(-x) = a\big(p(x)+p(-x)\big) = 0$. Hence, $ap(x) \in S$.
- Let $p(x)$ and $q(x)$ be polynomials belonging to the set S. Note that $p(x)+q(x)+p(-x)+q(-x) = p(x)+p(-x)+q(x)+q(-x) = 0$. Thus, $p(x)+q(x) \in S$.

So, we can conclude that S is a subspace of P_4. Note that $p(x)=a+bx+cx^2+dx^3$ belongs to S if and only if $p(x)+p(-x)=2a+2cx^2=0$, that is $a=c=0$. Therefore, $p(x)$ belongs to S if and only if it is of the form $p(x)=bx+dx^3$. Thus, we can conclude that the vectors x, x^3 form a basis for S and hence its dimension is 2.

4d: In order to show that L is a linear transformation, observe that:

- $L(ap(x)) = \frac{1}{2}\big(ap(x)+ap(-x)\big) = \frac{a}{2}\big(p(x)+p(-x)\big) = aL(p(x))$, and
- $L(p(x)+q(x)) = \frac{1}{2}\big(p(x)+q(x)+p(-x)+q(-x)\big) = \frac{1}{2}\big(p(x)+p(-x)\big)+\frac{1}{2}\big(q(x)+q(-x)\big) = L(p(x))+L(q(x))$

for all scalars a and polynomials $p(x)$, $q(x)$. Therefore, L is a linear transformation.

4e: Recall that

$$\ker L = \{p(x) \in P_4 \mid L(p(x)) = 0\}.$$

Let $p(x)=a+bx+cx^2+dx^3$. Note that $p(x) \in \ker L$ if and only if

$$0 = L(p(x)) = \frac{1}{2}\big(p(x)+p(-x)\big) = \frac{1}{2}(a+bx+cx^2+dx^3+a-bx+cx^2-dx^3) = a+cx^2.$$

Then, we have $p(x) \in \ker L$ if and only if $a=c=0$. Consequently, we get

$$\ker L = \{bx+dx^3 \mid b \text{and } d \text{ are scalars}\}.$$

4f: To find the matrix representation, we proceed with finding the action of L on each basis vectors:

$$L(1+x) = \frac{1}{2}(1+x+1-x) = 1 = 1\cdot(1+x) - 1\cdot(x+x^2) + 1\cdot(x^2+x^3) - 1\cdot(x^3)$$
$$L(x+x^2) = \frac{1}{2}(x+x^2-x+x^2) = x^2 = 0\cdot(1+x) + 0\cdot(x+x^2) + 1\cdot(x^2+x^3) - 1\cdot(x^3)$$
$$L(x^2+x^3) = \frac{1}{2}(x^2+x^3+x^2-x^3) = x^2 = 0\cdot(1+x) + 0\cdot(x+x^2) + 1\cdot(x^2+x^3) - 1\cdot(x^3)$$
$$L(x^3) = \frac{1}{2}(x^3-x^3) = 0 = 0\cdot(1+x) + 0\cdot(x+x^2) + 0\cdot(x^2+x^3) + 0\cdot(x^3).$$

Therefore, we obtain the following matrix representation:

$$\begin{bmatrix} 1 & 0 & 0 & 0 \\ -1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 \\ -1 & -1 & -1 & 0 \end{bmatrix}.$$